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Sequences and $Ker(R[X_1,\dots,X_g] \to R[tI])$

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Dedicated to the memory of Professor Hideyuki Matsumura

Abstract

Let $I = (b_1,\dots,b_g)R$ $(g \geq 2)$ be an ideal in a Noetherian ring R, let K be the kernel of the natural homomorphism from $R_g = R[X_1,\dots,X_g]$ onto $\mathbf{S} = R[tI]$ (the restricted Rees ring of R with respect to I), and let $J = (\{b_iX_j - b_jX_i;\ 1 \leq i < j \leq g\})R_g$. Then the main results in this paper strengthen two known results in the literature by showing: if $b_1,\dots,b_g$ is a regular sequence, then $K = J$ and, for all $n \geq 1, Ass(R_g/J^n) = Ass(R_g/K)$; and, if $b_1,\dots,b_g$ is an asymptotic sequence, then $K_a = J_a$ and, for all $n \geq 1$, $Ass(R_g/(J^n)_a) = Ass(R_g/K_a) = \{P; P$ is a minimal prime divisor of $K\}$, where L_a denotes the integral closure of the ideal L. © 1997 Elsevier Science B.V.

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1. Introduction

With the notation of the abstract, Micali showed in [7, Lemma 2, p. 42] that if $b_1,\dots,b_g$ is a regular sequence, then $K = J$, and that the converse holds if R is an integral domain. Also, Rees showed in [12, (2.1)] that if $height((b_1,\dots,b_g)R) = g \geq 2$ and R is a quasi-unmixed local ring, then $K_a = J_a$ and $Ass(R_g/(J^n)_a) = Ass(R_g/K_a)$ for all $n \geq 1$. Now it is shown in [6, Lemma 5.3] that an ideal of the principal class (that is, an ideal of height g that can be generated by g elements) in a quasi-unmixed local ring is generated by an asymptotic sequence, so with this in mind, both results are concerned with $Ker(R_g \to R[tI])$ when I is generated by a sequence (a regular sequence, for

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Micali's theorem; an asymptotic sequence, for Rees' theorem). Therefore each of these results suggests a sharpened version of the other. Namely, Rees' result suggests that the conclusion for Micali's theorem should be $K=J$ and $Ass(R_g/J^n)=Ass(R_g/K)$ for all $n \geq 1$. And Micali's result suggests that the conclusion for Rees' theorem should be valid when $b_1,\dots,b_g$ is an asymptotic sequence in an arbitrary Noetherian ring and that (some form of) a converse should be true. The main theorems in this paper show that both such sharpened versions hold.

In Section 2 we mention some of the nice structure of $(R_g)_P$ and the nice behavior of J_P and K_P, where $P \in Ass(R_g/J^n)$ (for some $n \geq 1$) and $I \not\subseteq P\cap R$, and then we prove three results which are used to shorten the proofs of Theorem 3.2 and Theorem 4.3.

In Section 3 the emphasis is on strengthening the conclusion of Micali's theorem, and it is shown in Theorem 3.2 that if $b_1,\dots,b_g$ is a regular sequence, then $K=J$ and $Ass(R_g/J^n)=Ass(R_g/K)$ for all $n \geq 1$. Then a corollary shows that if R is Cohen–Macaulay (resp., an integral domain, a Cohen–Macaulay integral domain and I is a normal ideal), then the form ring of R_g with respect to J is Cohen–Macaulay (resp., an integral domain, an integrally closed Cohen–Macaulay integral domain), and if R is either Cohen–Macaulay or an integrally closed integral domain, then the restricted Rees ring of R_g with respect to J has the same property.

Finally, in Section 4, Rees' theorem is strengthened in Theorem 4.3 by showing that if I is generated by an asymptotic sequence in an arbitrary Noetherian ring, then $J_a = K_a = Rad(K)$ and $Ass(R_g/(J^n)_a) = Ass(R_g/K_a) = \{P; P$ is a minimal prime divisor of $K\}$ for all $n \geq 1$. Finally, a converse of this result is also proved.

2. Preliminaries on $Ker(R_g \longrightarrow R[tI])$

In this brief section we introduce the notation that will be used in the remainder of this paper, and then mention some of the nice structure of $(R_g)_P$ and the nice behavior of J_P and K_P, where $P \in Ass(R_g/K^n) \cup Ass(R_g/J^n)$ for some $n \geq 1$ and $I \not\subseteq P \cap R$. Since most of these results are known to experts, proofs will generally be omitted.

We begin by specifying the notation.

2.1. Notation. The following notation is fixed for the remainder of this paper: $b_1,\dots,b_g$ ($g \geq 2$) are elements in a Noetherian ring R, and $K = Ker(R_g \longrightarrow \mathbf{S})$, where $R_g = R[X_1,\dots,X_g]$ and $\mathbf{S} = R[tb_1,\dots,tb_g]$ is the *restricted Rees ring of R with respect to* $I = (b_1,\dots,b_g)R$. (Here we assume that K is the kernel of the natural homomorphism α from R_g onto $\mathbf{S}$, so α is the R -homomorphism such that $\alpha(X_i) = tb_i$ for $i = 1,\dots,g$.) Also, $w_{i,j} = b_iX_j - b_jX_i$ for $1 \leq i < j \leq g$, $J = (\{w_{i,j};\ 1 \leq i < j \leq g\})R_g$, and $H_j = (\{w_{i,j}, w_{j,k}; 1 \leq i < j, j < k \leq g\})R_g$ for $j = 1,\dots,g$. Finally, $z_1,\dots,z_h$ are the prime divisors of zero in R, ordered so that $z_1,\dots,z_d$ ($1 \leq d \leq h$) are the minimal prime divisors of zero, and if b_j is not nilpotent, then $A_j = (R_g)_{b_j}$.

The following remark summarizes several well-known and/or easily proved facts concerning K, J and the rings $A_j = (R_g)_{b_j}$.

2.2. Remark. With Notation 2.1, the following hold:

(2.2.1) If b_j is not nilpotent, then $H_jA_j = JA_j = KA_j$ is generated by the regular sequence $X_1 - (c_1/c_j)X_j, \ldots, X_{j-1} - (c_{j-1}/c_j)X_j, X_{j+1} - (c_{j+1}/c_j)X_j, \ldots, X_g - (c_g/c_j)X_j$, where c_i is the image of b_i in A_j.

(2.2.2) K has exactly h prime divisors, say $P_1, \ldots, P_h$, and they may be subscripted so that $P_1, \ldots, P_d$ are its minimal prime divisors and $P_j \cap R = z_j$ for $j = 1, \ldots, h$.

(2.2.3) If $P \in Ass(R_g/K^n) \cup Ass(R_g/J^n)$ for some $n \geq 1$ and if $I \not\subseteq z = P \cap R$, say $b_j \notin z$, then $z \in Ass(R)$ and for all $m \geq 1$ it holds that $P \in Ass(R_g/K^m) \cap Ass(R_g/J^m) \cap Ass(R_g/H_j^m)$.

(2.2.4) If $P_1, \ldots, P_d$ are as in (2.2.2) and if $I \not\subseteq P_1 \cup \ldots \cup P_d$, then $\{P_1, \ldots, P_d\} \subseteq Ass(R_g/J)$ and, with $T_j = (R_g)_{P_j}$ for $j = 1, \ldots, d$, it holds that T_j/z_jT_j is a regular local ring of altitude $g-1$ and $(J^n)_aT_j = (K^n)_aT_j = (P_j{}^n)_aT_j = (P_j{}^nT_j)_a = (P_j{}^n, z_j)T_j$ for all $n \geq 1$. (Here, G_a denotes the *integral closure* of an ideal G.)

In Remark 2.3 we prove three results that will help shorten the proofs of the main results in Sections 3 and 4.

2.3. Remark.

(2.3.1) Let $\mathbf{A} = \{P;\ P \in Ass(R_g/J^n)$ for some $n \geq 1\}$ and assume that $I \not\subseteq P$ for all $P \in \mathbf{A}$. Then, for all $n \geq 1$, $J^n = K^n$ and $Ass(R_g/K^n) = Ass(R_g/J^n) = Ass(R_g/K) = \mathbf{A} = \{P_1, \ldots, P_h\}$, where the P_i $(i = 1, \ldots, h)$ are as in (2.2.2).

(2.3.2) Let $\mathbf{B} = \{P;\ P \in Ass(R_g/(J^n)_a)$ for some $n \geq 1\}$ and assume that $I \not\subseteq P$ for all $P \in \mathbf{B}$. Then, for all $n \geq 1$, $(J^n)_a = (K^n)_a$ and $Ass(R_g/(K^n)_a) = Ass(R_g/(J^n)_a) = Ass(R_g/K_a) = \mathbf{B} = \{P_1, \ldots, P_d\}$, where the P_i $(i = 1, \ldots, d)$ are as in (2.2.2). (*Note*: By (4.1.1), $\mathbf{B} = \hat{A}^*(J)$ is the set of asymptotic prime divisors of J.)

(2.3.3) If $\mathbf{B}$ is as in (2.3.2) and if $I \not\subseteq P$ for all $P \in \mathbf{B}$, then $K_a = J_a = Rad(K)$.

Proof. For (2.3.1), fix $n \geq 1$ and note that if $I \not\subseteq \cup\{P;\ P \in Ass(R_g/J^n)\}$, then (2.2.3) shows that each P in $Ass(R_g/J^n)$ is in $Ass(R_g/K^m)$ for all $m \geq 1$. Therefore, $Ass(R_g/J^n) \subseteq Ass(R_g/K^n)$, so $K^n(R_g)_P = J^n(R_g)_P$ for all $P \in Ass(R_g/J^n)$ (by (2.2.1), since $(R_g)_P = (A_j)_{PA_j}$ for some $j = 1, \ldots, g$), so it follows that $K^n \subseteq J^n$. Therefore $K^n = J^n$, since the opposite inclusion is clear, hence it follows that if $I \not\subseteq P$ for all $P \in \mathbf{A}$, then for all $n \geq 1$ it holds that $K^n = J^n$ and $\mathbf{A} = Ass(R_g/J^n) = Ass(R_g/K^n)$. In particular, $\mathbf{A} = Ass(R_g/J) = Ass(R_g/K)$, and $Ass(R_g/K) = \{P_1, \ldots, P_h\}$ by (2.2.2).

For (2.3.2), it is shown in [11, (2.4)] that if $P \in Ass(R_g/(J^n)_a)$ for some $n \geq 1$, then $P \in Ass(R_g/(J^m)_a)$ for all $m \geq n$, so [6, Proposition 3.17] shows that $P \in Ass(R_g/J^m)$ for all large m. Therefore, if $I \not\subseteq P$, then $K^m(R_g)_P = J^m(R_g)_P$ for all $m \geq 1$ (by (2.2.1), since $(R_g)_P = (A_j)_{PA_j}$ for some $j = 1, \ldots, g$), hence $(K^m(R_g)_P)_a = (J^m(R_g)_P)_a$ for all $m \geq 1$. It follows from this that if, for some $n \geq 1$, $I \not\subseteq \bigcup\{P;\ P \in Ass(R_g/(J^n)_a)\}$, then $(K^n)_a \subseteq (\bigcap\{(K^n(R_g)_P)_a;\ P \in Ass(R_g/(J^n)_a)\}) \cap R_g = (\bigcap\{(J^n(R_g)_P)_a;\ P \in Ass(R_g/(J^n)_a)\}) \cap R_g = (J^n)_a$. Therefore $(K^n)_a = (J^n)_a$, since the opposite inclusion is clear, so it follows that if $I \not\subseteq P$ for all $P \in \mathbf{B}$, then $(K^n)_a = (J^n)_a$ for all $n \geq 1$, hence $Ass(R_g/(K^n)_a) = Ass(R_g/(J^n)_a)$ for all $n \geq 1$. Also, it follows from (2.2.2) that each P_i $(i = 1,\dots,d)$ is a minimal prime divisor of K, so each is in $Ass(R_g/(K^n)_a)$ for all $n \geq 1$, so it follows that $\{P_1,\dots,P_d\} \subseteq Ass(R_g/(K^n)_a) = Ass(R_g/(J^n)_a) \subseteq \mathbf{B}$. Therefore, to complete the proof it must be shown that $\mathbf{B} \subseteq \{P_1,\dots,P_d\}$.

For this, let $P \in \mathbf{B}$ and assume that $I \not\subseteq P$. Then [11, (2.4)] shows that $P \in Ass(R_g/(J^n)_a)$ for all large n, so [6, Proposition 3.18] shows that there exists a minimal prime ideal z in R_g such that $z \subseteq P$ and $P/z \in Ass((R_g/z)/(((J/z)^n)_a))$ for all large n, hence $P/z \in Ass((R_g/z)/((J/z)^n))$ for all large n, by [6, Proposition 3.17]. Also, $z = (z \cap R)R_g$ and $z \cap R$ is a minimal prime ideal, and it is readily checked that $J/z = (\{\overline{b_i}X_j - \overline{b_j}X_i;\ 1 \leq i < j \leq g\})(R/z)_g$, where the "bar" denotes residue class modulo z. Further, $\overline{I} \not\subseteq P/z$ (since $I \not\subseteq P$, by hypothesis), so it follows from (2.2.3) that $P/z \in Ass((R_g/z)/Q)$, where $Q = Ker((R/z)_g \to (\overline{R}[t\overline{I}]))$, and Q is prime, since $\overline{R}[t\overline{I}]$ is an integral domain, hence $Q = P/z$. Therefore $(P/z) \cap (R/(z \cap R)) = (0)$, since $Q \cap (R/(z \cap R)) = (0)$, so it follows that: $(*)$ $P \cap R = z \cap R$ is a minimal prime ideal. Also, $P \in Ass(R_g/(J^n)_a)$ for all large n, as noted above, so [6, Proposition 3.17] shows that $P \in Ass(R_g/J^n)$ for all large n, and $I \not\subseteq P$, by hypothesis, so $P \in Ass(R_g/K)$, by (2.2.3). Therefore, since $P \cap R = z \cap R$ is a minimal prime ideal (by $(*)$), it follows from (2.2.2) that $P \in \{P_1,\dots,P_d\}$. Therefore, it follows that if $I \not\subseteq P$ for all $P \in \mathbf{B}$, then $\mathbf{B} \subseteq \{P_1,\dots,P_d\}$.

Finally, for (2.3.3), if $I \not\subseteq P$ for all $P \in \mathbf{B}$, then (2.3.2) shows that $J_a = K_a$ and that $Ass(R_g/K_a) = \{P_1,\dots,P_d\}$. It then follows from (2.2.4) that $K_a = \bigcap\{(K_a(R_g)_{P_j}) \cap R_g;\ j = 1,\dots,d\} = \bigcap\{(P_j(R_g)_{P_j}) \cap R_g;\ j = 1,\dots,d\} = Rad(K)$. □

3. Regular sequences and $Ker(R_g \to R[tI])$

Let $b_1,\dots,b_g$ be a regular sequence in a Noetherian ring R. Then a fairly self-contained proof of the fact that $K = J$ is given in [7, Lemma 2, p. 42], but we do not know how to use it to show that then $Ass(R_g/J^n) = Ass(R_g/K)$ for all $n \geq 1$. So in this section we give in Theorem 3.2 a new proof (using generically perfect ideals (see (3.1.2))) that $K = J$, and this approach yields the additional conclusion concerning the prime divisors of J^n. Then this section is closed by proving a useful corollary.

To prove Theorem 3.2 we need the following definitions.

3.1. Definition.

(3.1.1) Let R be a Noetherian ring and let J be a proper ideal of R. Then J is said to be *perfect* in case $grade(J) = proj.dim._R(R/J)$.

(3.1.2) Let S be a polynomial ring over $\mathbb{Z}$, the integers, and let J be a homogeneous perfect ideal in S. Then J is *generically perfect* in case S/J is faithfully flat over $\mathbb{Z}$ (equivalently, since J is homogeneous, torsion-free over $\mathbb{Z}$).

References for generically perfect ideals include [1–4]. By the main result of [2], if J is a generically perfect homogeneous ideal and one replaces the variables in S

by a regular sequence contained in the Jacobson radical of a Noetherian ring, then the resulting ideal is also perfect.

3.2. Theorem. *Let $I = (b_1,\ldots,b_g)R$, R_g, K, and J be as in Notation 2.1 and assume that $b_1,\ldots,b_g$ form a regular sequence. Then:*

(3.2.1) $J = K$.

(3.2.2) $Ass(R_g/J^n) = Ass(R_g/K)$ *for all* $n \geq 1$.

Proof. By (2.3.1) it suffices to show that $I \not\subseteq P \cap R$ for all $P \in \mathbf{A} = \{P;\ P \in Ass(R_g/J^n)$ for some $n \geq 1\}$. For this, note that if $P \in \mathbf{A}$, then there exists $p \in Ass(R_g[tJ]/JR_g[tJ])$ such that $p \cap R_g = P$. (For if $\mathbf{R} = R_g[t^{-1}, tJ]$ is the Rees ring of R_g with respect to J, then $t^{-n}\mathbf{R} \cap R_g = J^n$, so there exists a prime divisor q of $t^{-n}\mathbf{R}$ such that $q \cap R_g = P$. Then q is a prime divisor of $t^{-1}\mathbf{R}$, and $\mathbf{R}/t^{-1}\mathbf{R} = R_g[tJ]/JR_g[tJ] = \mathbf{F}(R_g,J)$ (the form ring of R_g with respect to J), so $p = q \cap R_g[tJ]$ is a prime divisor of $JR_g[tJ]$ such that $p \cap R_g = P$.) Therefore, if we can show that $I \not\subseteq p$ for all $p \in Ass(R_g[tJ]/JR_g[tJ]))$, then we will be done.

For this, let $W_{i,j}$ $(1 \leq i < j \leq g)$ be $\binom{g}{2}$ indeterminates and map $R_g[\{W_{i,j}\}]$ onto $\mathbf{F}(R_g,J)$ by sending $W_{i,j}$ to the J -form $w^*_{i,j}$ of $w_{i,j}$. Call this map ϕ, so ϕ is the map presenting $\mathbf{F}(R_g,J)$. We are going to prove that $I \not\subseteq p$ for all $p \in Ass(R_g[tJ]/JR_g[tJ])$ by using the fact that the kernel of ϕ is a generically perfect ideal in the case where $b_1,\ldots,b_g$ are indeterminates.

To elaborate, let $B_1,\ldots,B_g$, $X_1,\ldots,X_g$, and $\{W_{i,j}\}$ be indeterminates over $\mathbb{Z}$, the integers, and set $w'_{i,j} = B_iX_j - B_jX_i$, for $1 \leq i < j \leq g$. Set $R' = \mathbb{Z}[B_1,\ldots,B_g]$, $R'_g = R'[X_1,\ldots,X_g]$, and $J' = (\{w'_{i,j}\}) \cdot R'_g$, and let $Q' \subseteq R'_g[\{W_{i,j}\}]$ be the kernel of the map presenting the form ring $\mathbf{F}(R'_g,J')$ of R'_g with respect to J'. Then [1, Theorems 9.14 and 9.17] show that $\mathbf{F}(R'_g,J')$ is an integrally closed Cohen–Macaulay integral domain. It therefore follows that Q' is a perfect prime ideal of grade $\binom{g}{2}$ (see [1, Proposition 16.19]). Thus, $R'_g[\{W_{i,j}\}]/Q'$ is $\mathbb{Z}$ torsion-free, so Q' is generically perfect. Let Q denote the image of Q' in $R_g[\{W_{i,j}\}]$ obtained by setting $B_i = b_i$, for $i = 1,\ldots,g$. We now show that Q is a perfect ideal of grade $\binom{g}{2}$.

To see that Q is perfect, we may check by localizing $R_g[\{W_{i,j}\}]$ at any homogeneous maximal ideal $\mathcal{M}$ containing Q. In other words, if we show that

$$grade(Q_{\mathcal{M}}) = proj.dim._{R_g[\{W_{i,j}\}]_{\mathcal{M}}}(R_g[\{W_{i,j}\}]_{\mathcal{M}}/Q_{\mathcal{M}}) = \binom{g}{2}$$

for every homogeneous maximal ideal $\mathcal{M}$, then we obtain that Q is perfect with $\text{grade}(Q) = \binom{g}{2}$. For this, $\mathcal{M} = (M,X_1,\ldots,X_g,\{W_{i,j}\})R_g[\{W_{i,j}\}]$ for some maximal ideal M in R. If $I \not\subseteq \mathcal{M}$, then the proof of the first claim below shows that $Q_{\mathcal{M}}$ is generated by a regular sequence of length $\binom{g}{2}$. Therefore, we may assume that $I \subseteq M$. Then $Q_{\mathcal{M}}$ is obtained from a generically perfect ideal (namely, Q') by replacing the variables $B_1,\ldots,B_g,X_1,\ldots,X_g,\{W_{i,j}\}$ by the regular sequence $b_1,\ldots,b_g,X_1,\ldots,X_g,\{W_{i,j}\}$ contained in the Jacobson radical of $R_g[\{W_{i,j}\}]_{\mathcal{M}}$, and therefore, $Q_{\mathcal{M}}$ is a perfect ideal of grade $\binom{g}{2}$ by the Hochster–Eagon theorem mentioned following (3.1.2).

We now make two claims. The first claim is that $Q_{b_1} = (Ker(\phi))_{b_1}$, and the second claim is that no prime divisor of Q contains b_1. If both claims hold, then (since it is clear that $Q \subseteq Ker(\phi)$) it follows that $Q = Ker(\phi)$ and that no prime divisor of $Ker(\phi)$ contains I, so no prime divisor of zero in $\mathbf{F}(R_g, J)$ contains I, hence $I \nsubseteq p$ for all $p \in Ass(R_g[tJ]/JR_g[tJ])$, which is what we want to prove.

For the first claim, note that $b_1 w_{j,k} = b_j w_{1,k} - b_k w_{1,j}$ $(1 < j < k \leq g)$, so it follows that the $\binom{g-1}{2}$ elements $W_{j,k} - (b_j/b_1)W_{1,k} + (b_k/b_1)W_{1,j}$ $(1 < j < k \leq g)$ are in $(Ker(\phi))_{b_1}$. Also, $J_{b_1} \subseteq (Ker(\phi))_{b_1}$, and (2.2.1) shows that $K_{b_1} = J_{b_1} = (H_1)_{b_1}$ is generated by the $w_{1,i}$ $(1 < i \leq g)$ (which generate a regular sequence of length $g-1$ in $R_g[1/b_1]$). Therefore, the form ring of $R_g[1/b_1]$ with respect to $K_{b_1} = J_{b_1} = (H_1)_{b_1}$ is a polynomial ring in $g - 1$ variables over $R_g[1/b_1]/K_{b_1} = R_g[1/b_1]/H_1R_g[1/b_1]$. It follows at once that the elements $W_{j,k} - (b_j/b_1)W_{1,k} + (b_k/b_1)W_{1,j}$ $(1 < j < k \leq g)$, and $w_{1,i}$ $(1 < i \leq g)$ generate $(Ker(\phi))_{b_1}$. Since these elements clearly belong to Q_{b_1}, and since $Q_{b_1} \subseteq (Ker(\phi))_{b_1}$, we must have $Q_{b_1} = (Ker(\phi))_{b_1}$. Therefore the first claim holds.

It now remains to see that no prime divisor of Q contains b_1. For this, suppose, on the contrary, that $b_1 \in \mathbf{p}$ for some prime divisor $\mathbf{p}$ of Q. Now grade($\mathbf{p}$) $= \binom{g}{2}$, since Q is perfect. In fact, [1, Proposition 16.17] shows that

(3.2.3) $grade(\mathbf{p}_\mathbf{p}) = \binom{g}{2}$.

We will now obtain a contradiction by showing that, since $b_1 \in \mathbf{p}$, $grade(\mathbf{p}_\mathbf{p}) \geq \binom{g}{2} + 1$.

For this, since $J \subseteq Q$ and $b_1 \in \mathbf{p}$, it follows that either: (a) $I \subseteq \mathbf{p}$; or, (b) $I \nsubseteq \mathbf{p}$ and $X_1 \in \mathbf{p}$. If (a) holds, then $\mathbf{p}$ contains the regular sequence $b_1, \ldots, b_g$ of length g. Therefore at least one $W_{i,j}$, say $W_{1,2}$, is not in $\mathbf{p}$. Now $\mathbf{p}$ contains the $g - 2$ elements $X_k W_{1,2} - X_2 W_{1,k} + X_1 W_{2,k}$ $(3 \leq k \leq g)$, and the $\binom{g-2}{2}$ Plücker relations $W_{1,2}W_{j,k} - W_{1,j}W_{2,k} + W_{1,k}W_{2,j}$ $(3 \leq j < k \leq g)$. If we localize at $\mathbf{p}$, then $W_{1,2}$ becomes a unit, so these elements taken together yield a regular sequence of length $g + (g-2) + \binom{g-2}{2} = \binom{g}{2} + 1$ contained in $\mathbf{p}_\mathbf{p}$, and this contradicts (3.2.3), so (a) does not hold. (Of course, we also obtain a similar contradiction assuming any other $W_{i,j} \notin \mathbf{p}$.)

If (b) holds, then without loss of generality we may assume that $b_2 \notin \mathbf{p}$. Then $\mathbf{p}$ contains b_1, X_1 and the $g - 2$ elements $w_{2,j}$ $(3 \leq j \leq g)$. Additionally, $\mathbf{p}$ contains the $\binom{g-2}{2}$ elements $b_2 W_{j,k} - b_j W_{2,k} + b_k W_{2,j}$ $(3 \leq j < k \leq g)$ and the $g - 2$ elements $b_1 W_{2,j} - b_2 W_{1,j} + b_j W_{1,2}$ $(3 \leq j \leq g)$. If we localize at $\mathbf{p}$, then b_2 becomes a unit, so these elements taken together yield a regular sequence of length $\binom{g}{2} + 1$ contained in $\mathbf{p}_\mathbf{p}$, and this contradicts (3.2.3), so (b) does not hold. Therefore, we have a contradiction to the supposition that $b_1 \in \mathbf{p}$, since neither (a) nor (b) holds, so this contradiction completes the proof of the second claim, hence the theorem holds. □

Corollary 3.3. is a corollary of Theorem 3.2 (and its proof). It shows that $R_g[tJ]$, the restricted Rees ring of R_g with respect to J, and $\mathbf{F}(R_g, J)$, the form ring of R_g with respect to J, inherit several nice properties from R when $b_1, \ldots, b_g$ is a regular sequence. (The reader should note that each statement in Corollary 3.3 holds when $b_1, \ldots, b_g$ are indeterminates (see [1, Ch. 9]). Moreover, though Corollary 3.3 is clearly related to [1, Propositions 3.11–3.13], it does not seem to follow immediately from them.)

3.3. Corollary. *Assume that R is a Noetherian ring and $I = (b_1,\dots,b_g)R$ is an ideal in R such that* $\text{grade}(I) = g \geq 2$. *Then the following hold:*

(3.3.1) *If R is Cohen–Macaulay (resp., an integral domain, a Cohen–Macaulay integral domain and I is a normal ideal (that is, I^n is integrally closed for all positive integers n)), then $\mathbf{F}(R_g,J) = \mathbf{F}(R_g,K)$ is Cohen–Macaulay (resp., an integral domain, an integrally closed Cohen–Macaulay integral domain).*

(3.3.2) *If R is Cohen–Macaulay (resp., an integrally closed integral domain), then $R_g[tJ] = R_g[tK]$ is Cohen–Macaulay (resp., an integrally closed integral domain).*

Proof. Since an ideal of grade g that is generated by g elements can be generated by a regular sequence, and since the form ring and Rees ring of an ideal are independent of the generating set, we may assume that $b_1,\dots,b_g$ is an R -sequence. Then Theorem 3.2 shows that $J = K$, so it suffices to prove the results concerning $\mathbf{F}(R_g,J)$ (in (3.3.1)) and $R_g[tJ]$ (in (3.3.2)).

For (3.3.1) assume that R is an integral domain. Then $\mathbf{S}$ is, so $J = K$ is a prime ideal. Moreover, J_{b_1} is generated by a regular sequence, so it follows that $\mathbf{F}(R_g,J)_{\overline{b_1}}$ is an integral domain, where $\overline{b_1}$ denotes the image of b_1 in $\mathbf{F}(R_g,J)$. Also, the two claims in the proof of Theorem 3.2 show that $\overline{b_1}$ is regular in $\mathbf{F}(R_g,J)$, so it follows that $\mathbf{F}(R_g,J)$ is an integral domain.

Next assume that R is Cohen–Macaulay. Then $R_g[\{W_{i,j}\}]$ is Cohen–Macaulay and the proof of Theorem 3.2 shows that $\mathbf{F}(R_g,J) = R_g[\{W_{i,j}\}]/Q$, where Q is a perfect ideal and $grade(Q) = \binom{g}{2}$. Therefore it follows that $\mathbf{F}(R_g,J)$ is a Cohen–Macaulay ring (see [1, Proposition 16.19]).

Finally, assume that R is a Cohen–Macaulay integral domain and that I is a normal ideal. Then R_g/J is integrally closed, $(R_g)_J$ is a regular local ring, and $\mathbf{F}(R_g,J)$ is a Cohen–Macaulay integral domain (by what has already been proved), so it remains to show that $\mathbf{F}(R_g,J)$ is integrally closed. For this, by [5, Corollary 2.1], it suffices to show that $l(J_q) \leq \max\{dim(R_g)_J, dim(R_g)_q - 2\}$, for all prime ideals q of R_g that contain J. (Here, $l(J_q)$ denotes the analytic spread of J_q. Note also that the assumption in [5, Corollary 2.1] that R be a homomorphic image of a regular ring is not required.) However, $l(J_q) \leq 2g - 2$ for all such prime ideals q (as in the proof of [12, (2.1)]), and if $b_i \notin q$ for some i or if $X_j \notin q$ for some j, then $l(J_q) = g - 1$. Therefore it follows that $\mathbf{F}(R_g,J)$ is integrally closed.

For (3.3.2), we may argue as in the proof of Theorem 3.2 that the kernel of the map presenting $R_g[tJ]$ is a perfect ideal of grade $\binom{g}{2} - 1$. It then follows from this as in the proof of (3.3.1), that $R_g[tJ]$ is Cohen–Macaulay whenever R is Cohen–Macaulay. If R is an integral domain, then $J = K$ is a prime ideal, so $J^n = K^n$ is K-primary for all $n \geq 1$, by Theorem 3.2. Therefore $J_K = K_K$ is a normal ideal, so J is a normal ideal. Since R is also integrally closed, $R_g[tJ]$ is integrally closed. □

4. Asymptotic sequences and $Ker(R_g \to R[tI])$

The main result in this section, Theorem 4.3, extends Rees' theorem concerning K and J to arbitrary asymptotic sequences (of length $g \geq 2$) in arbitrary Noetherian rings. To prove Theorem 4.3 we need the following definitions and preliminary result (which is of some interest in itself).

4.1. Definition. Let I be an ideal in a Noetherian ring R.

(4.1.1) $\hat{A}^*(I) = \{P \in Spec(R);\ P$ is a prime divisor of $(I^n)_a$ for all large n (equivalently, by [R3, (2.4)], for some $n \geq 1$)$\}$ is the set of *asymptotic prime divisors* of I. Here $(I^n)_a$ is the *integral closure* in R of I^n. (Therefore $\hat{A}^*(J) = \mathbf{B}$ of (2.3.2).)

(4.1.2) An element b in R is *asymptotically prime* to I in case $(I,b)R \neq R$ and $(I^n)_a : bR = (I^n)_a$ for all $n \geq 1$. Elements $b_1,\ldots,b_g$ in R are an *asymptotic sequence* in R in case b_i is asymptotically prime to $(b_1,\ldots,b_{i-1})R$ for $i = 1,\ldots,g$. (In particular, since $(0)_a = Rad(R)$, it follows that b_1 is not in any minimal prime ideal in R.)

Concerning (4.1.2), it is shown in [6, Lemma 5.13] that a regular sequence is an asymptotic sequence, so the results in this section hold when $b_1,\ldots,b_g$ is a regular sequence in a Noetherian ring R.

The following result shows a useful new property of asymptotic sequences (namely, they become a regular sequence in many complete local domains that are closely related to the original ring). (It follows quite directly from the Cohen structure theorems that if C is a coefficient subring of a complete local domain L and $x_1,\ldots,x_m$ is a system of parameters in L, then $D = C[[x_1,\ldots,x_m]]$ is a complete local subdomain of L, L is a finite D -module, D is a complete intersection, and $x_1,\ldots,x_m$ is a regular sequence in D.)

4.2. Proposition. *Let $b_1,\ldots,b_g$ be an asymptotic sequence in a Noetherian ring R, let P be a prime ideal in R that contains $b_1,\ldots,b_g$, let $Q = R_P$, let z be a minimal prime ideal in the completion Q^* of Q, let $L = Q^*/z$, let C be a coefficient subring of L, and for $i = 1,\ldots,g$ let x_i be the image of b_i in L. Then altitude$(L) = m \geq g$ and there exist $x_{g+1},\ldots,x_m$ in L such that:*

(4.2.1) *$C[[x_1,\ldots,x_m]]$ is a complete local subdomain of L.*

(4.2.2) *L is a finite $C[[x_1,\ldots,x_m]]$-module.*

(4.2.3) *$C[[x_1,\ldots,x_m]]$ is a complete intersection.*

(4.2.4) *$x_1,\ldots,x_m$ is a regular sequence in $C[[x_1,\ldots,x_m]]$.*

Proof. It is shown in [6, Remark (b), p. 32] that the images in Q of $b_1,\ldots,b_g$ form an asymptotic sequence, so they form an asymptotic sequence in Q^*, by [6, Lemma 5.1], so their images $x_1,\ldots,x_g$ form a subset of a system of parameters in $L = Q^*/z$, by [6, Lemmas 5.1 and 5.2]. Therefore by extending $x_1,\ldots,x_g$ to a system of parameters $x_1,\ldots,x_g,x_{g+1},\ldots,x_m$ of L the conclusions follow immediately from the comment preceding this proposition. □

The next result, which is an asymptotic sequence version of Theorem 3.2, extends the main result in [12] by showing that the conclusion holds for an arbitrary asymptotic sequence (of length $g \geq 2$) in an arbitrary Noetherian ring. As mentioned in the introduction, Rees proved the theorem for asymptotic sequences in a quasi-unmixed local ring in [12, (2.1)], but since Theorem 3.2 did not require that R be Cohen–Macaulay, an asymptotic sequence version should not require that R be quasi-unmixed (since, in the correspondence between the asymptotic and standard theories of ideals, quasi-unmixed is the analog of Cohen–Macaulay), and, in fact, Theorem 4.3 verifies this. (Rees' proof in [12] is essentially self-contained (and is quite pretty). In contrast, the proof given below uses (2.3.2), (2.3.3), Theorem 3.2, and Proposition 4.2, and several results on asymptotic sequences that have previously appeared in the literature.)

4.3. Theorem. *Let $b_1, \ldots, b_g$ $(g \geq 2)$ be an asymptotic sequence in a Noetherian ring R and let R_g, K, J, and* **S** *be as in Notation* 2.1. *Then*:

(4.3.1) $J_a = K_a = Rad(K)$, *so* $(J^n)_a = (K^n)_a$ *for all* $n \geq 1$.

(4.3.2) *For all* $n \geq 1$, $Ass(R_g/(J^n)_a) = Ass(R_g/K_a)$ *is the set of minimal prime divisors of* K.

Proof. Assume it is known that: $(*)$ $I \not\subseteq P \cap R$ for all $P \in \hat{A}^*(J)$. Then the conclusions follow immediately from (2.3.3) and (2.3.2). Therefore, it remains to show that $(*)$ holds.

For this, suppose, on the contrary, that there exists $P \in \hat{A}^*(J)$ such that $I \subseteq P \cap R$. Let $p = P \cap R$ and let $Q = R_p$. Then $PQ_g \in \hat{A}^*(JQ_g)$ by [6, Remark, p. 15] (since Q_g is a localization of R_g), and $IQ \subseteq PQ_g \cap Q$. Let Q^* be the completion of Q. Then $(Q^*)_g$ is a faithfully flat extension ring of Q_g, so by [11, (6.5)] there exists $P^* \in \hat{A}^*(J(Q^*)_g)$ such that $P^* \cap Q_g = PQ_g$, so $IQ^* \subseteq P^* \cap Q^*$. Therefore there exists a minimal prime ideal z contained in P^* such that, with $L = Q^*/(z \cap Q^*)$, $P^*/z \in \hat{A}^*(JL_g)$ by [6, Proposition 3.18], and $IL \subseteq (P^*/z) \cap L$. Finally, the images $b_1', \ldots, b_g'$ in L of $b_1, \ldots, b_g$ are a subset of a system of parameters (by [6, Remark, p. 15, and Lemmas 5.1 and 5.2]), so let $altitude(L) = g + e$ and let $b_{g+1}', \ldots, b_{g+e}'$ in L such that $b_1', \ldots, b_{g+e}'$ is a system of parameters in L, let C be a coefficient subring of L, and let $D = C[[b_1', \ldots, b_{g+e}']]$. Then $N = (P^*/z) \cap D_g \in \hat{A}^*(JD_g)$ by [6, Proposition 3.22] (since it follows from (4.2.3) that L_g is a finite integral extension domain of D_g), and $(b_1', \ldots, b_g')D \subseteq N \cap D$. However, $b_1', \ldots, b_g'$ is a regular sequence in D, by (4.2.4), so Theorem 3.2 shows that $\hat{A}^*(JD_g)$ is a one point set whose one element contracts in D to zero (since $\mathbf{S}(D, (b_1', \ldots, b_g')D)$ is an integral domain); hence we have a contradiction. Therefore $(*)$ holds. □

It is well known that if $b_1, \ldots, b_g$ are contained in the Jacobson radical of R, then they form a regular sequence if and only if $K \subseteq IR_g$ if and only if $J = K$. Theorem 4.4, which is a converse of Theorem 4.3, gives an asymptotic analog of this result.

4.4. Theorem. *Let $I = (b_1,\dots,b_g)R$ be an ideal of the principal class ($g \geq 2$) that is contained in the Jacobson radical of a Noetherian ring R and for each $n \geq 1$, let $J(n) = (\{b_i{}^n X_j - b_j{}^n X_i;\ 1 \leq i < j \leq g\})R_g$ and $K(n) = Ker(R_g \to \mathbf{S}(n))$, where $\mathbf{S}(n) = R[tb_1{}^n, tb_2{}^n, \dots, tb_g{}^n]$. Then the following are equivalent:*

(4.4.1) *$b_1,\dots,b_g$ is an asymptotic sequence in R.*

(4.4.2) *$K(n) \subseteq (I^n)_a R_g$ for all $n \geq 1$.*

(4.4.3) *$(J(n))_a = (K(n))_a$ for all $n \geq 1$.*

Proof. If (4.4.1) holds, then $b_1{}^n,\dots,b_g{}^n$ is an asymptotic sequence in R for all $n \geq 1$, by [10, (3.15)], so (4.4.1) $\Rightarrow$ (4.4.3) by (4.3.1). And since $J(n) \subseteq I^n R_g$ for all $n \geq 1$, it follows that $(J(n))_a \subseteq (I^n R_g)_a = (I^n)_a R_g$, so (4.4.3) $\Rightarrow$ (4.4.2).

Finally, let $\mathbf{R}(n) = R[u, tb_1{}^n, \dots, tb_g{}^n]$, so $\mathbf{R}(n)/u\mathbf{R}(n) = \mathbf{S}(n)/I^{[n]}\mathbf{S}(n) = \mathbf{F}(n)$ where $\mathbf{F}(n)$ is the form ring of R with respect to $I^{[n]} = (b_1{}^n, b_2{}^n, \dots, b_g{}^n)R$. Then it is shown in [9, (4.17)(1′) $\Leftrightarrow$ (4′)] that (4.4.1) is equivalent to $\mathbf{R}(n)/(u,(I^n)_a)\mathbf{R}(n) = R_g/(I^n)_a R_g$ for all $n \geq 1$. Therefore, if (4.4.2) holds, then since $(I^{[n]})_a = (I^n)_a$ it follows that $R_g/(I^n)_a R_g = \mathbf{S}(n)/(I^n)_a\mathbf{S}(n) = \mathbf{F}(n)/((I^n)_a/I^{[n]})\mathbf{F}(n) = \mathbf{R}(n)/(u,(I^n)_a)\mathbf{R}(n)$, so (4.4.2) $\Rightarrow$ (4.4.1). □

The final result is a corollary of Theorem 4.4, and it is an asymptotic sequence version of the following well known characterization of a Cohen–Macaulay local ring: The following are equivalent for a local ring R such that $altitude(R) = n \geq 2$: (a) R is Cohen–Macaulay. (b) There exists a system of parameters $b_1,\dots,b_n$ in R such that $Ker(R_n \to R[tb_1,\dots,tb_n]) \subseteq (b_1,\dots,b_n)R_n$. (c) For every ideal $I = (b_1,\dots,b_g)R$ of the principal class $g \geq 2$ it holds that $Ker(R_g \to R[tb_1,\dots,tb_g]) \subseteq IR_g$.

4.5. Corollary. *The following are equivalent for a local ring R such that $altitude(R) = n \geq 2$:*

(4.5.1) *R is quasi-unmixed.*

(4.5.2) *There exists a system of parameters $b_1,\dots,b_n$ in R such that $Ker(R_n \to R[tb_1{}^m,\dots,tb_n{}^m]) \subseteq ((b_1{}^m,\dots,b_n{}^m)R)_a)R_n$ for all $m \geq 1$.*

(4.5.3) *For every ideal $I = (b_1,\dots,b_g)R$ of the principal class $g \geq 2$ it holds that $Ker(R_g \to R[tb_1,\dots,tb_g]) \subseteq I_a R_g$.*

Proof. (4.5.1) $\Rightarrow$ (4.5.3) by (4.4.1) $\Rightarrow$ (4.4.2) (since ideals of the principal class in a quasi-unmixed local ring are generated by an asymptotic sequence, by [6, Lemma 5.3]), it is clear that (4.5.3) $\Rightarrow$ (4.5.2), and (4.5.2) $\Rightarrow$ (4.5.1) by (4.4.2) $\Rightarrow$ (4.4.1) (since a local ring with an open ideal generated by an asymptotic sequence is quasi-unmixed, by [6, Corollary 5.9]). □

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